Hierarchical and evolved large-scale network structures

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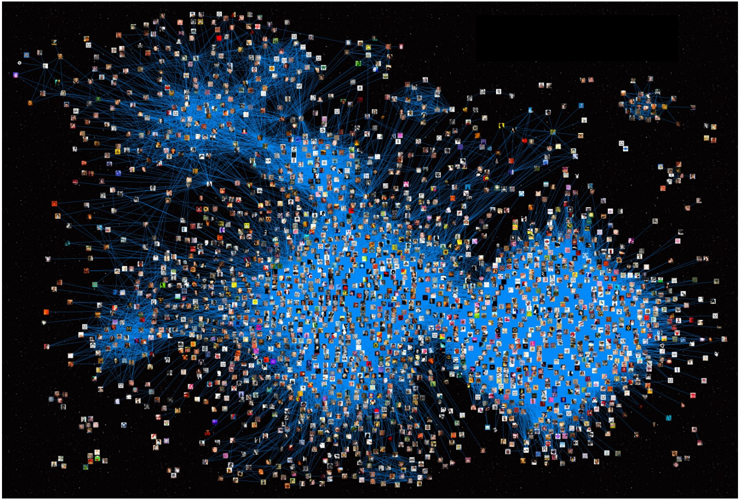
Universität Bremen

Berkeley, June 2014

Global (or large-scale) structure

Modular structure



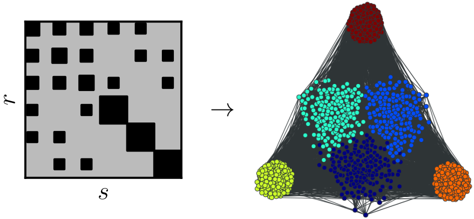
A PRINCIPLED APPROACH: GENERATIVE MODELS

STATISTICAL INFERENCE OF **STOCHASTIC BLOCK MODELS**

Traditional: N nodes divided into B blocks.

Parameters: $b_i \to$ block membership of node i

$e_{rs} \to$ number of edges from block r to s.



Degree-corrected: Arbitrary degree sequence: $\{k_i\}$

P. W. Holland et al., Soc Networks 5, 109 (1983)
B. Karrer and M. Newman, Phys. Rev. E 83, 016107 (2011)

Generative models

Two complementary approaches:

Inference of model parameters from empirical data

Abstract modelling of network function

Direct bridge between data and functional models, which incorporate large-scale network topology.

Inference via maximum likelihood

Microcanonical formulation: $\mathcal{P}(G|\{e_{rs}\},\{b_i\}) = \dfrac{1}{\Omega(\{e_{rs}\},\{b_i\})}$

T.P.P., PRE 85, 056122 (2012)

Inference via maximum likelihood

Microcanonical formulation: $\mathcal{P}(G|\{e_{rs}\},\{b_i\}) = \dfrac{1}{\Omega(\{e_{rs}\},\{b_i\})}$

Ensemble entropy: $\mathcal{S}(\{e_{rs}\},\{b_i\}) = \ln\Omega(\{e_{rs}\},\{b_i\}) = -\ln\mathcal{P}(G|\{e_{rs}\},\{b_i\})$

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$$\mathcal{S} \cong -E - \sum_k N_k \ln k! - \frac{1}{2}\sum_{rs} e_{rs} \ln\left(\frac{e_{rs}}{e_r e_s}\right)$$

Inference $\leftrightarrow$ Compression

$$\max_{\{e_{rs}\},\{b_i\}} \ln\mathcal{P} \equiv \min_{\{e_{rs}\},\{b_i\}} \mathcal{S}$$

$\rightarrow$ Minimization of information required to describe the network, *when the model is known*.

T.P.P., PRE 85, 056122 (2012)

Solution: Minimum Description Length

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$$\mathcal{L} = \underbrace{\ln \left(\!\!\binom{\left(\!\binom{B}{2}\!\right)}{E}\!\!\right)}_{\text{Block graph}} + \underbrace{\ln \left(\!\!\binom{B}{N}\!\!\right) + \ln N! - \sum_r \ln n_r!}_{\text{Node partition}} + \underbrace{\sum_r n_r H(\{p_k^r\})}_{\text{Degree sequence}}$$

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$$\boxed{\Sigma = \mathcal{S} + \mathcal{L}}$$

Total information necessary, without a priori knowledge of the model!

T.P.P., Phys. Rev. Lett. 110, 148701 (2013)

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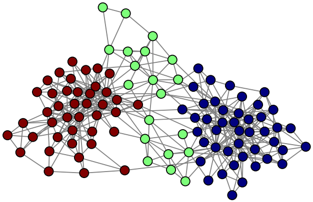
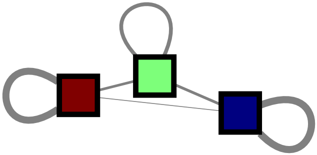
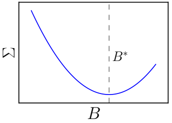




$B = 3$, $\mathcal{S} \simeq 1688.1$ bits

Model, $\mathcal{L} \simeq 203.4$ bits

$\Sigma \simeq 1891.6$ bits



$\rightarrow$

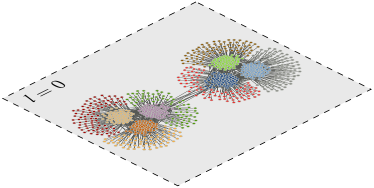
Occam's razor

The best model is the one which most compresses the data.

T.P.P., Phys. Rev. Lett. 110, 148701 (2013)

Problem: Lack of prior information

Solution → model the model



T.P.P., Phys. Rev. X 4, 011047 (2014)

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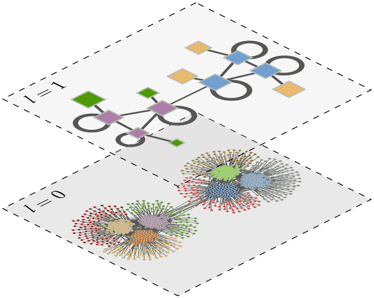

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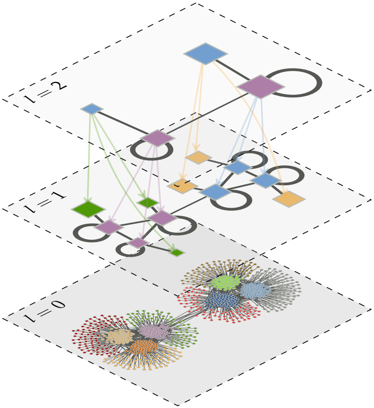

T.P.P., Phys. Rev. X 4, 011047 (2014)

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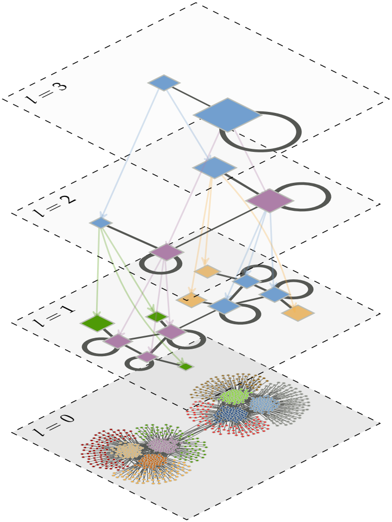

T.P.P., Phys. Rev. X 4, 011047 (2014)

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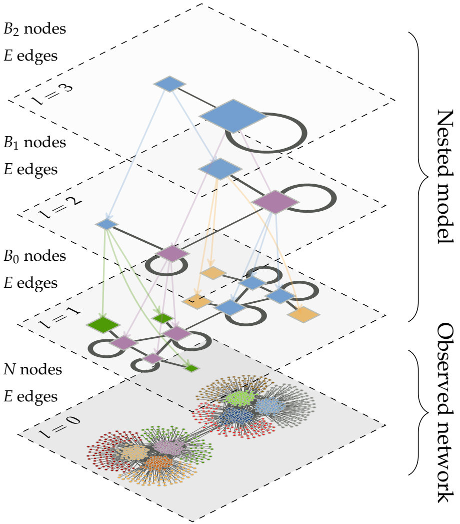

T.P.P., Phys. Rev. X 4, 011047 (2014)

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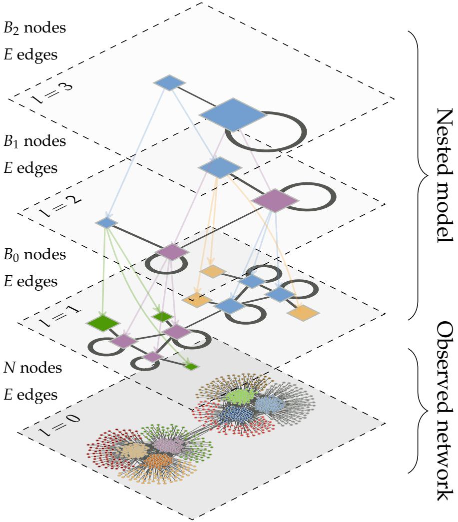
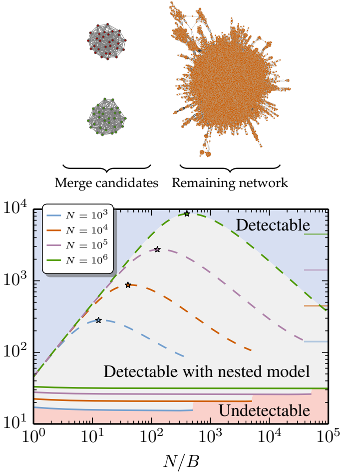


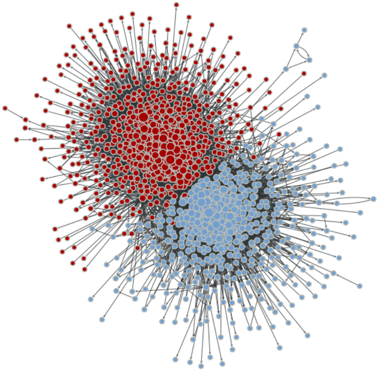


$$N/B_{\max} \sim \ln N$$

T.P.P., Phys. Rev. X 4, 011047 (2014)

EMPIRICAL NETWORKS

POLITICAL BLOGS ($N = 1,222, E = 19,027$)



T.P.P., Phys. Rev. X 4, 011047 (2014)

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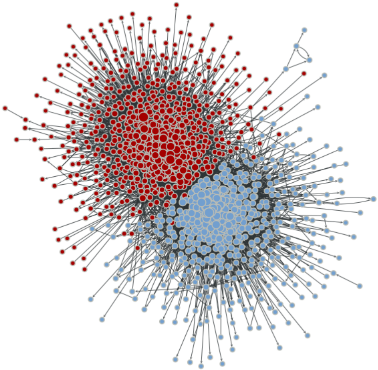
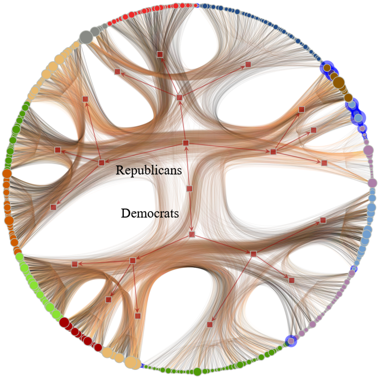
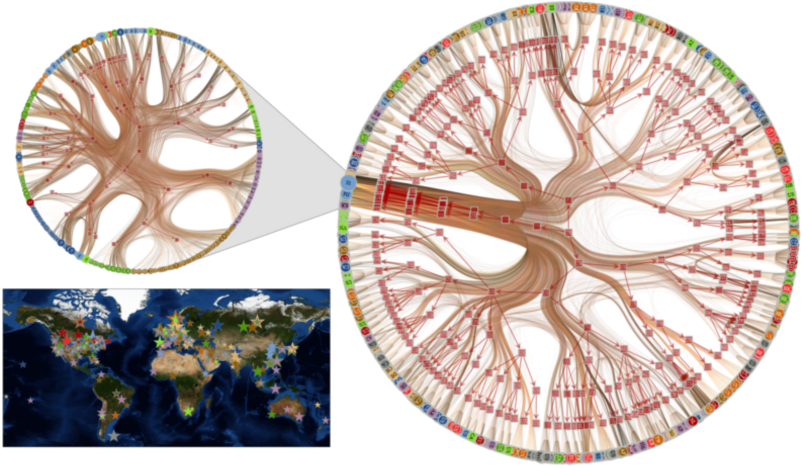




T.P.P., Phys. Rev. X 4, 011047 (2014)

Empirical networks

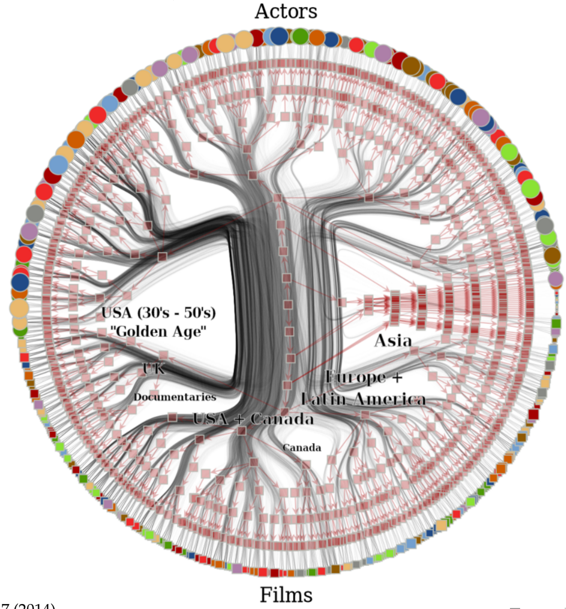
Internet (Autonomous systems) ($N = 52{,}104, E = 399{,}625, B = 191$)



T.P.P., Phys. Rev. X 4, 011047 (2014)

Empirical networks

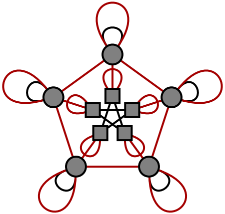
IMDB Film-Actor Network ($N = 372,447, E = 1,812,312, B = 717$)

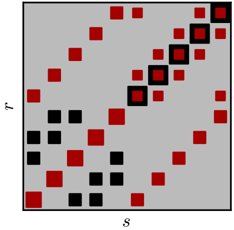


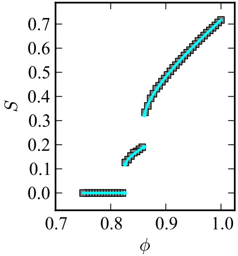
T.P.P., Phys. Rev. X 4, 011047 (2014)

ABSTRACT MODELLING

EXAMPLE: INTERDEPENDENT PERCOLATION







$$u_r = \sum_s m_{rs} \left[1 - \hat{\phi}_s f_1^s(1) + \hat{\phi}_s f_1^s(u_s)\right]$$

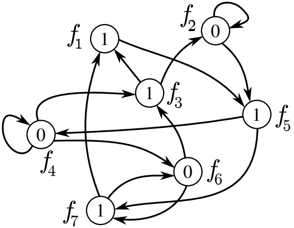
$$\hat{\phi}_r = 1 - \hat{f}_0^r(1 - \textstyle\sum_s \hat{m}_{rs} S_s^0) + \hat{f}_0^r(0)$$

Generalization of two-interdependent networks, “network of networks”, etc.

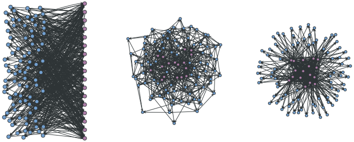
T. P. P., S. Bornholdt, PRL 109, 118703 (2012)

Abstract modelling

Example: Noisy Boolean dynamics

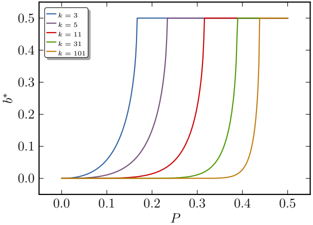


σ_4	σ_6	f_3
0	0	1
0	1	0
1	0	0
1	1	1



(a) (b) (c)

$$b_i(t+1) = \sum_k p_k^i m_k \left((1-2P) \sum_j w_{j\to i} b_j(t) + P \right)$$



How would networks look like if they are optimized according to some criterion, *but are otherwise maximally random*?

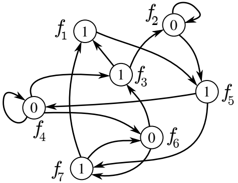
Minimization of the free energy:

$$\boxed{\mathcal{F} = -R - S/\beta}$$

$R \rightarrow$ Fitness criterion
$\beta \rightarrow$ Fitness pressure

Robustness criteria

1. Dynamical: Boolean dynamics with noise



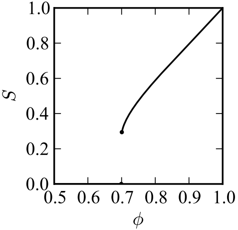
σ_4	σ_6	f_3
0	0	1
0	1	0
1	0	0
1	1	1

$$\sigma_i(t+1) = f_i\left(\boldsymbol{\sigma}(t)\right)$$

Noise P

$\boxed{R \rightarrow \text{fraction of “correct” nodes}}$

2. Structural: Percolation with interdependence



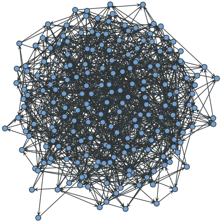
$$\boxed{R = 2\int_0^1 S(\phi)d\phi}$$

$$R \in [0,1]$$

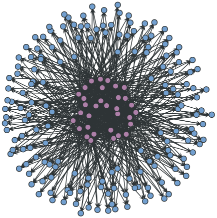
$S \rightarrow$ size of macroscopic component
$\phi \rightarrow$ dilution (fraction of nodes not removed)

Dynamical robustness against noise

Boolean networks and gene regulation

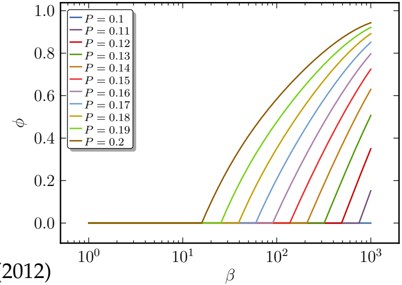


increasing β



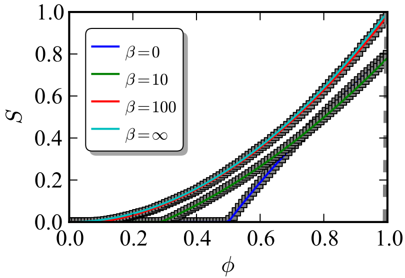
Random topology

Segregated core



T. P. Peixoto, PRE 85, 041908 (2012)

Percolation: Optimization



T. P. P., S. Bornholdt, PRL 109, 118703 (2012)

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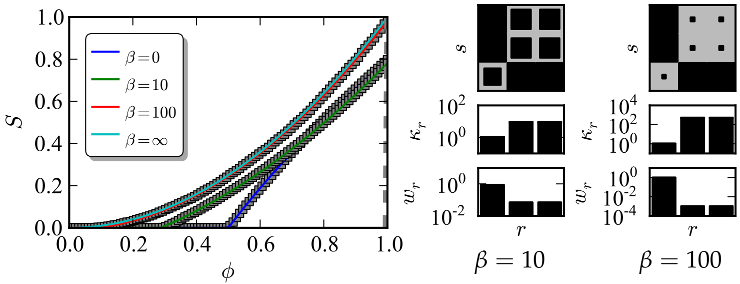

T. P. P., S. Bornholdt, PRL 109, 118703 (2012)

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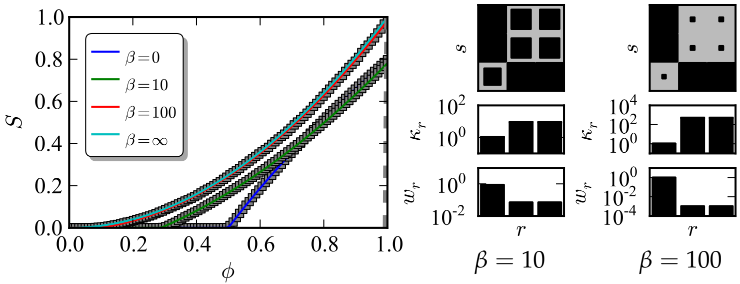
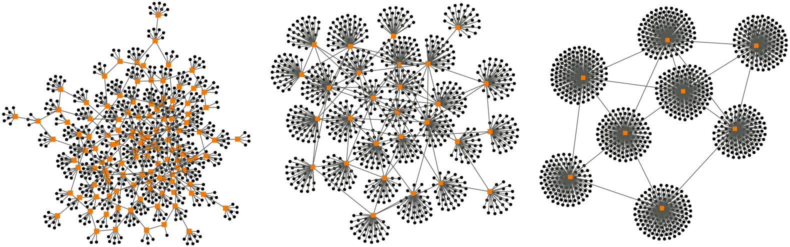


- Core-periphery structure!
- Independent of the number of blocks, B!

T. P. P., S. Bornholdt, PRL 109, 118703 (2012)

Percolation: Optimization



$\beta = 10$ $\beta = 20$ $\beta = 40$

Backbones...

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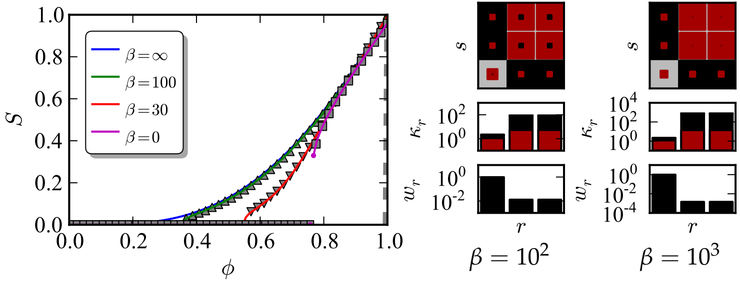
Percolation: Optimization

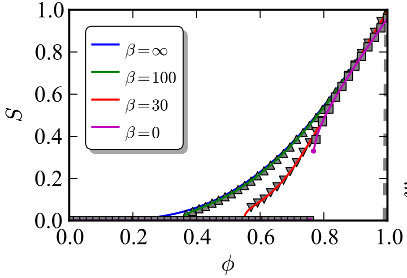
With interdependence

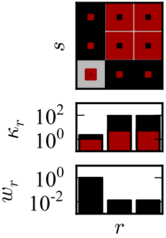


T. P. P., S. Bornholdt, PRL 109, 118703 (2012)

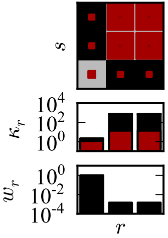
Percolation: Optimization

With interdependence

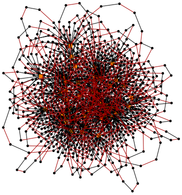




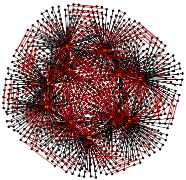
$\beta = 10^2$



$\beta = 10^3$



$\beta = 10$



$\beta = 30$

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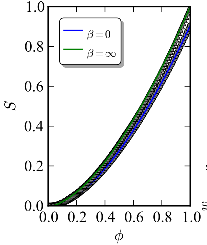
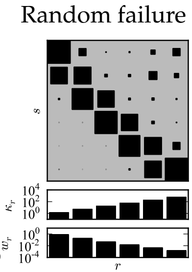
Constrained optimization

Degree constraints



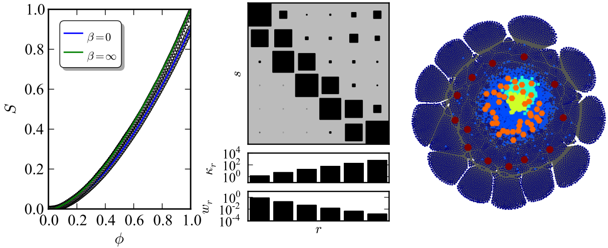


T. P. P., S. Bornholdt, PRL 109, 118703 (2012)

Constrained optimization

Degree constraints

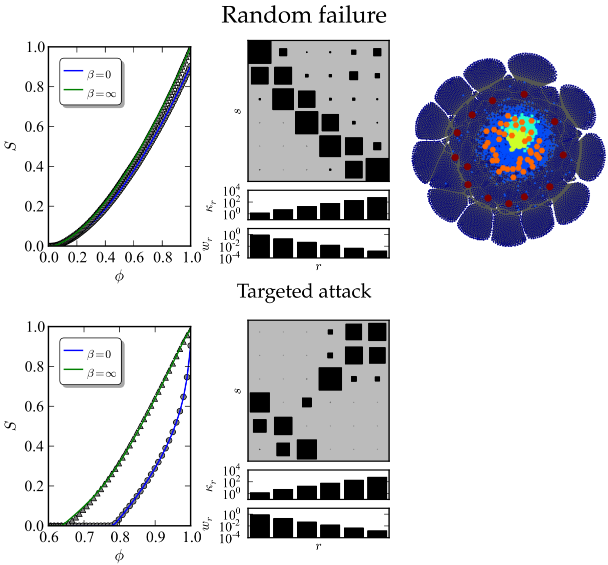
Random failure



T. P. P., S. Bornholdt, PRL 109, 118703 (2012)

Constrained optimization

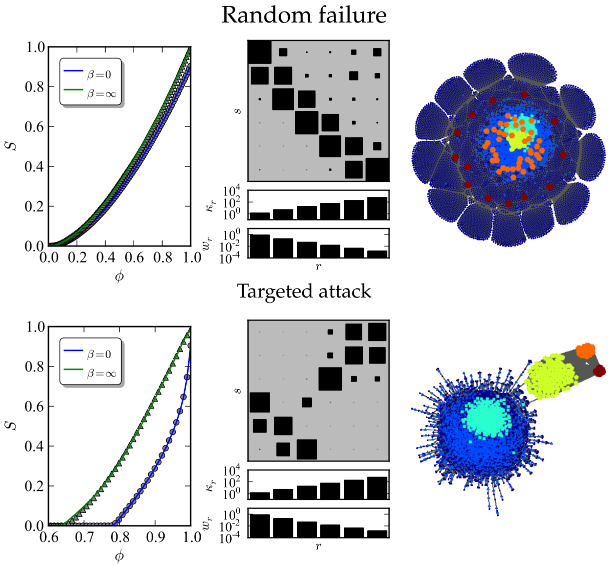
Degree constraints



T. P. P., S. Bornholdt, PRL 109, 118703 (2012)

CONSTRAINED OPTIMIZATION

DEGREE CONSTRAINTS



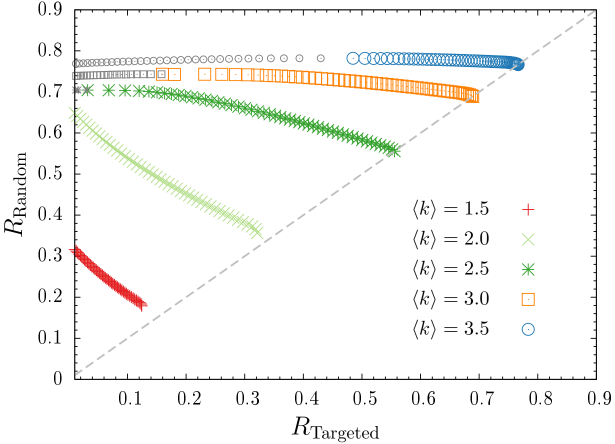
"Onionlike" topology
Schneider et at.
PNAS 2011

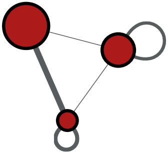
T. P. P., S. Bornholdt, PRL 109, 118703 (2012)

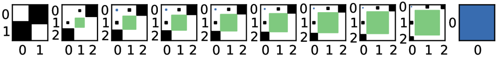
Multi-objective optimization

Random failure vs. targeted attacks

Pareto-optimal fronts







C. Priester, S. Schmitt, T.P.P., submitted.

Conclusion

- Stochastic block models (SBM) $\rightarrow$ simple, tractable, general
- Principled inference of large-scale structure of empirical data
- Nested SBM $\rightarrow$ multilevel network structures, high resolution
- SBMs $\rightarrow$ convenient for abstract modelling
- Maximum entropy SBM ensembles: Null models of robust topologies
- Robust topologies are simple!
- Emergence of a core-periphery structure, with a central well-connected backbone, and a sparse periphery.
- More elaborate structures arise when arbitrary constraints or competing criteria are imposed.