Structure of complex networks: Quantifying edge-to-edge relations by failure induced flow redistribution

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Higher Order Models in Network Science Satellite meeting

Michael T. Schaub
Department of Mathematics
Imperial College London

J. Lehmann, S. N. Yaliraki and M. Barahona

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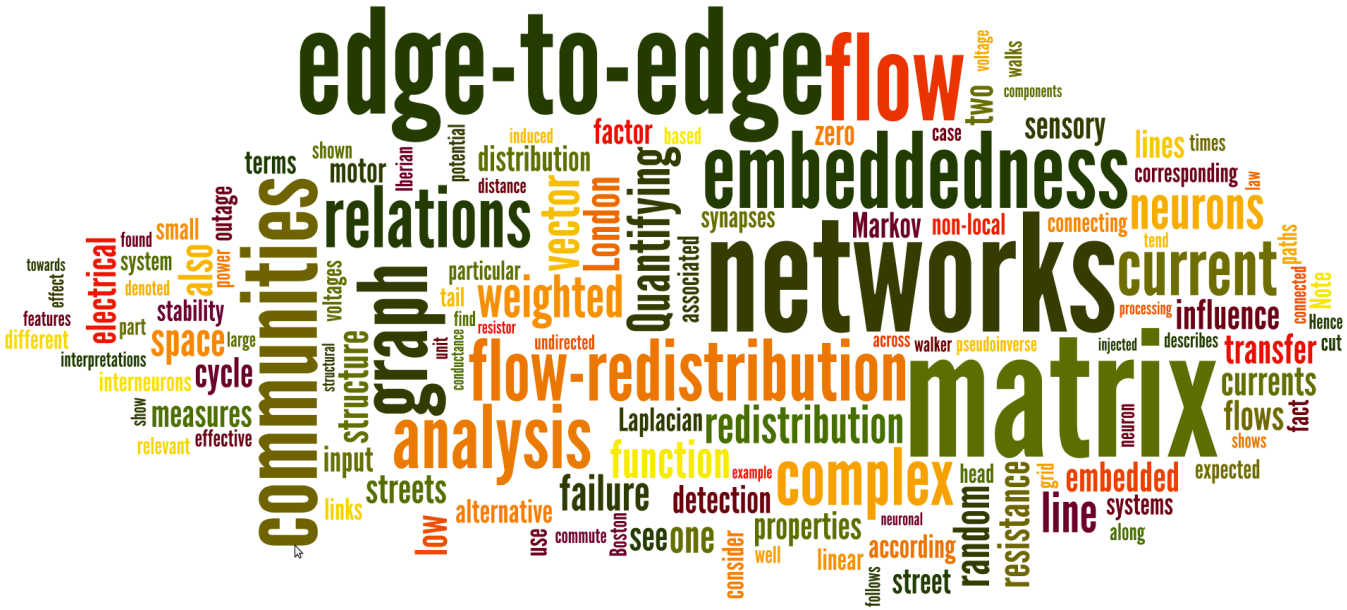
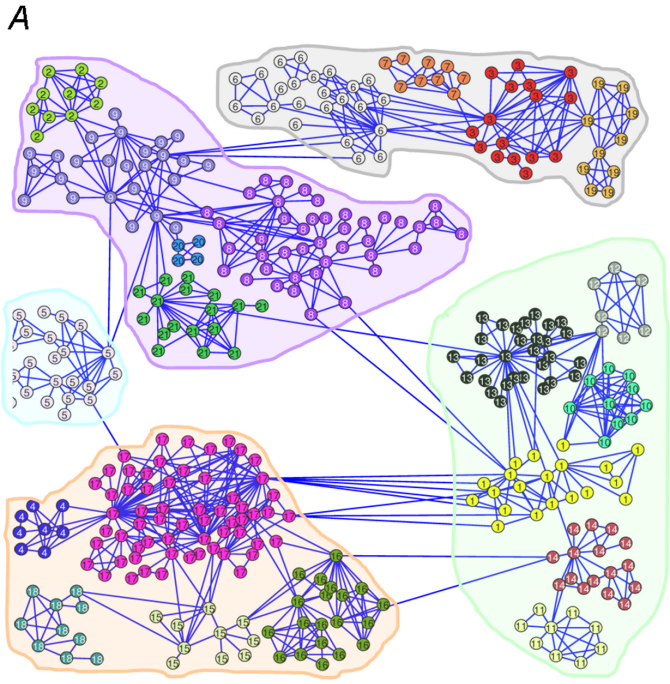
Outline

- Background: why edge relations?
- Edge relations through flow redistribution
- Using edge relations for network analysis

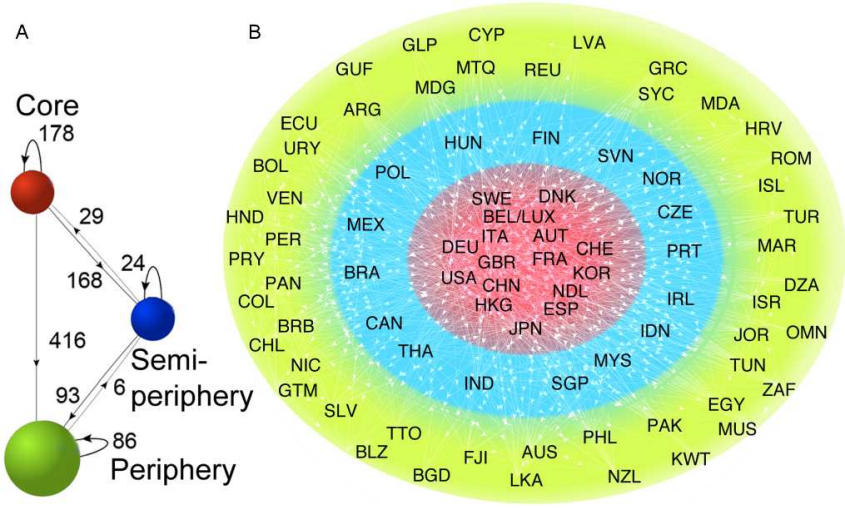


Background: why edge relations?

- Network Analysis so far mainly **node** centric
- Communities, node roles, centralities, etc.

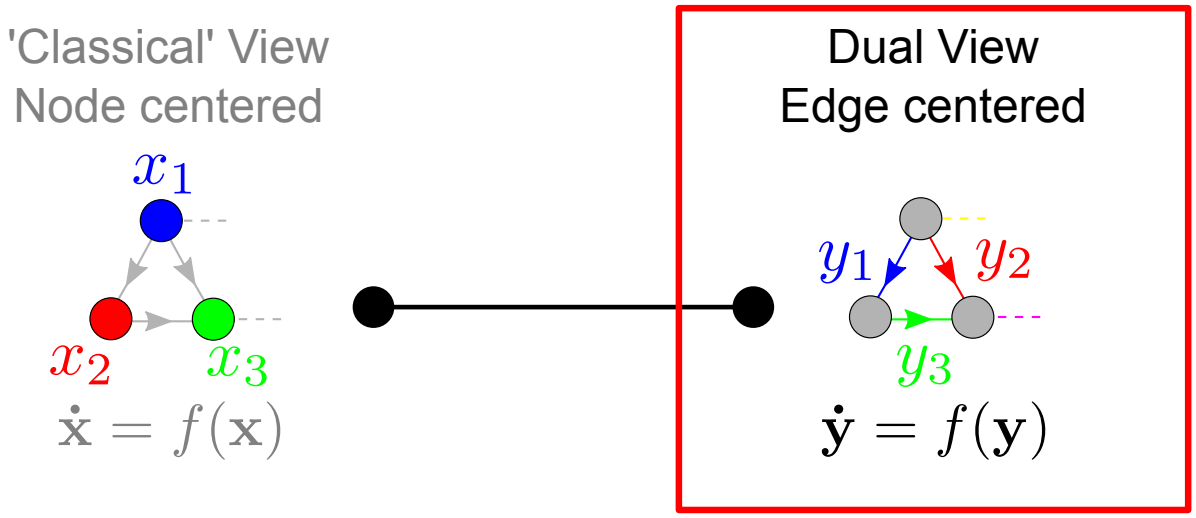


[Delvenne et al., 2010]



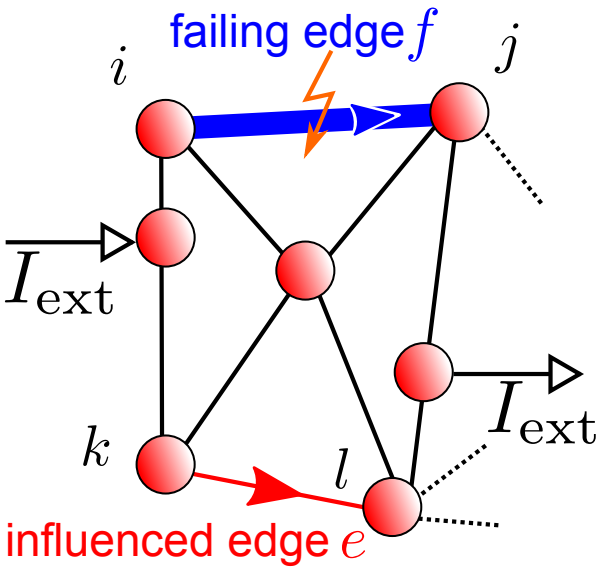
[Cooper et al 2010]

Dual Perspective – Edge centered



- Circuit Theory: voltage *vs* currents
- Computational mechanics: displacement *vs* stress
- Optimization: Primal *vs* Dual variables
- Systems engineering, estimation theory, etc...

How to quantify edge relations? Flow redistribution!



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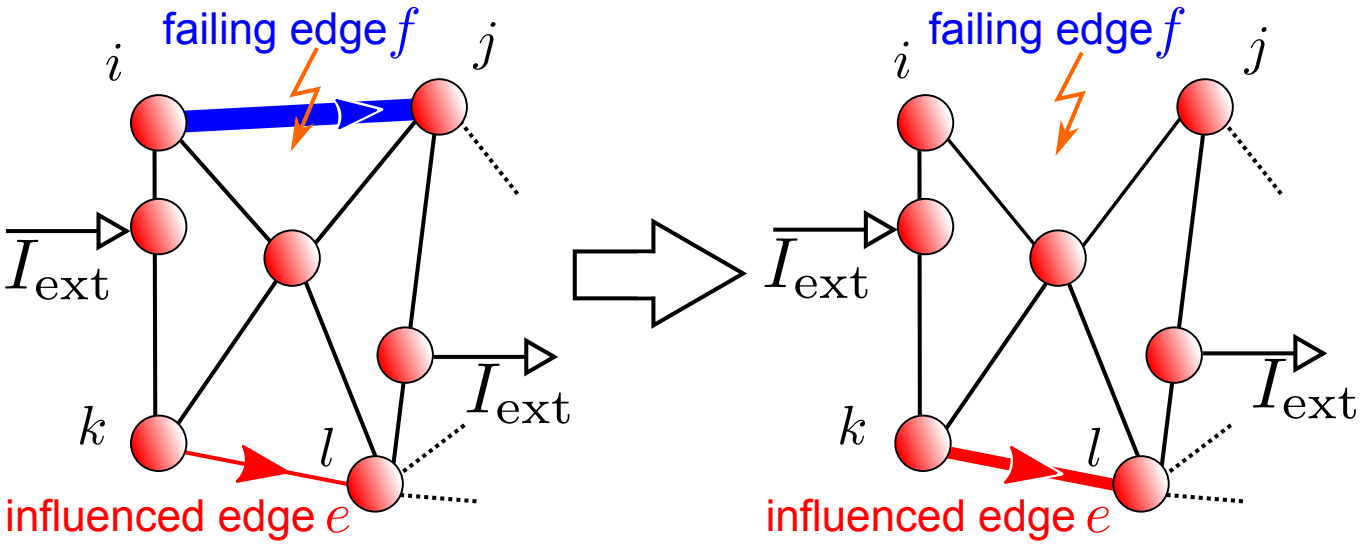

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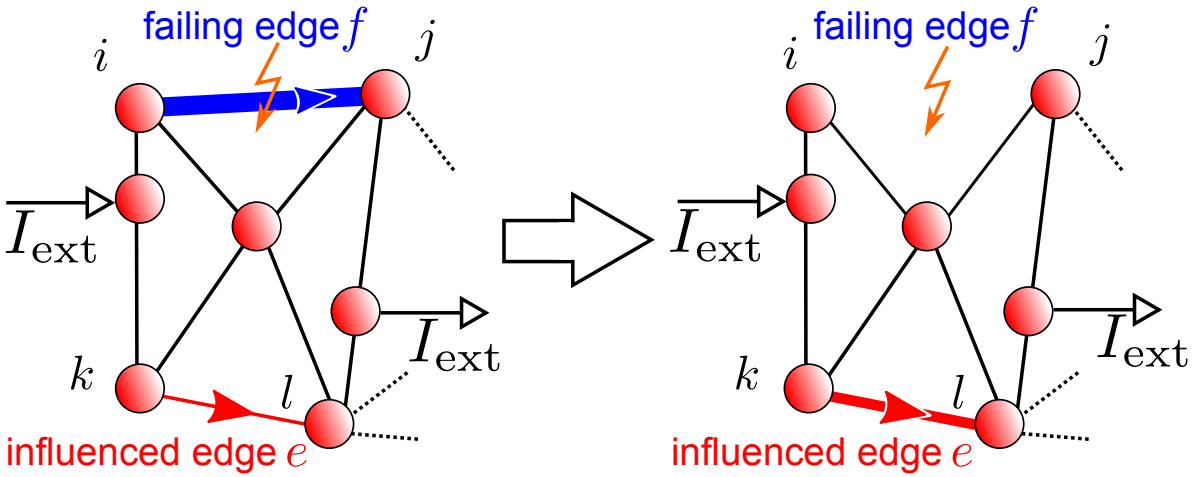
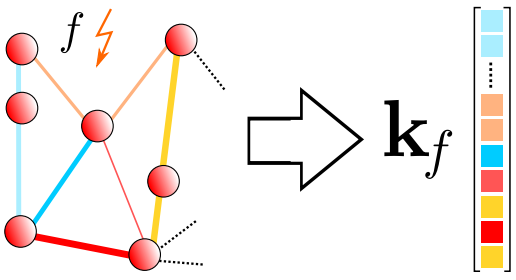


Assuming a linear flow on the edges



$$\mathbf{k}_f$$

$$\Delta_f \mathbf{i} = \left[\frac{G B^T L^\dagger \mathbf{b}_f}{1 - g_f \, \mathbf{b}_f^T L^\dagger \mathbf{b}_f} \right] \mathrm{i}_f \equiv \mathbf{k}_f \; \mathrm{i}_f$$

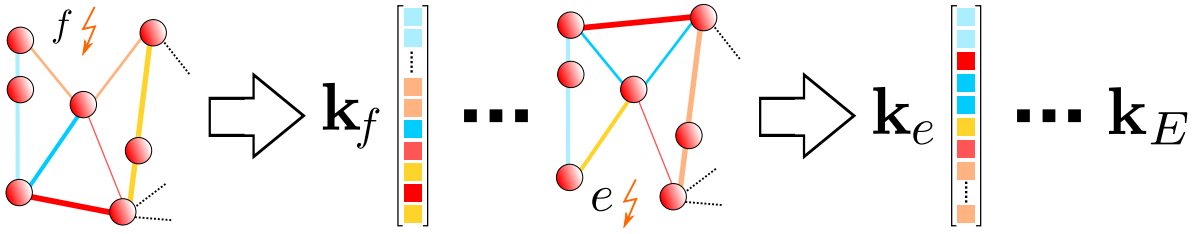
G=diag(g) — edge weight

incidence matrix — incidence vector

pseudoinverse of Laplacian

independent of current injections

The flow redistribution matrix



- Flow redistribution matrix

$$K_{E\times E} \equiv [\mathbf{k}_1 \cdots \mathbf{k}_E]$$

- Independent of current injections
- Describes **topological** feature of system in the edge space: **edge-to-edge coupling**

Characterising the flow redistribution matrix

The flow redistribution matrix can be decomposed

$$K = M\left[\mathrm{diag}(\boldsymbol{\varepsilon})\right]^{-1}$$

into the edge-to-edge transfer function matrix

$$M_{E\times E} \equiv G B^T L^\dagger B$$

and the edge-embeddedness

$$\varepsilon_e \equiv 1 - g_e\,\mathbf{b}_e^T L^\dagger \mathbf{b}_e = 1 - g_e R_e$$

The edge to edge transfer function

$$M_{E\times E} \equiv GB^T L^\dagger B$$

- **Transfer function** – describes how input on edge translates into flow on other edges
- Physics interpretation – **discrete Green's function** (edge space)
- **Projection matrix** (idempotent) – into the weighted cut space of the graph
  - spectral properties of M

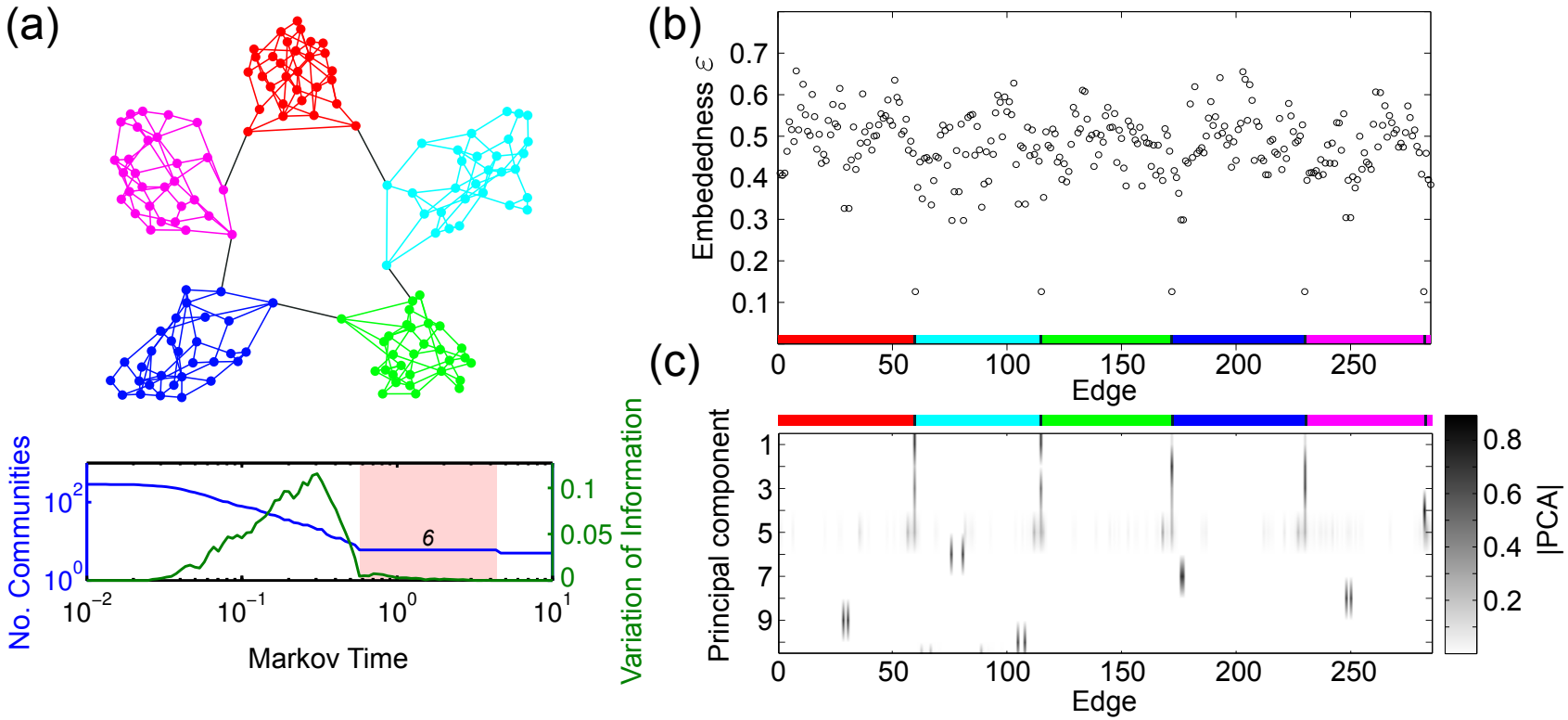
The edge-embeddedness

$$\varepsilon_e \equiv 1 - g_e \, \mathbf{b}_e^T L^\dagger \mathbf{b}_e = 1 - g_e R_e$$

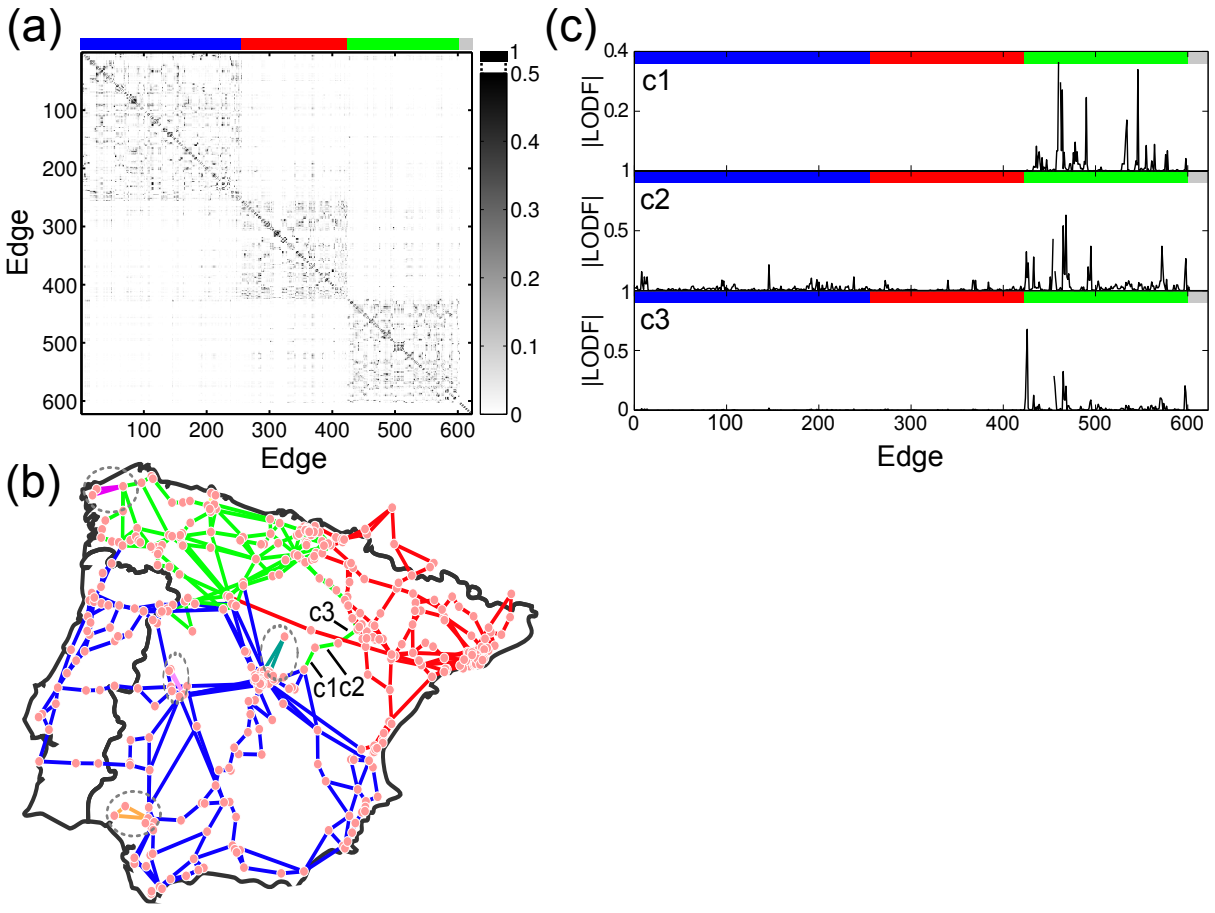
R_e – resistance distance between endpoints of edge e

- related to the projection into the **cycle space**
- High embeddedness – edge features in many cycles (weighted)
- Zero embeddedness – edge defines a cut (disconnects the network)
- Unweighted networks – probability of *not* finding the edge in a randomly selected spanning tree
- $\sum \varepsilon_e =$ #cycles in network
- Related to **graph sparsification**

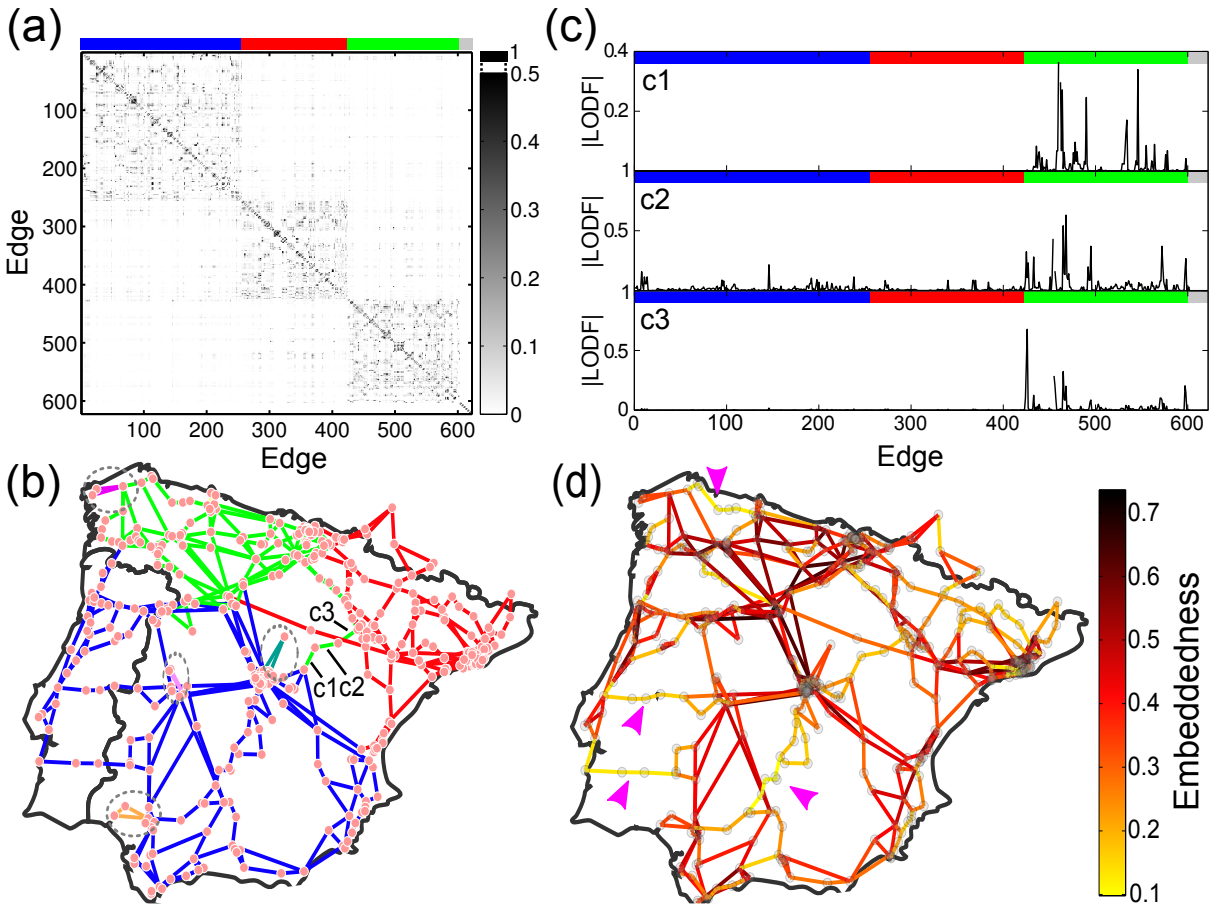
Applications: a toy example



Applications 1 – Iberian Power Grid



Applications 1 – Iberian Power Grid



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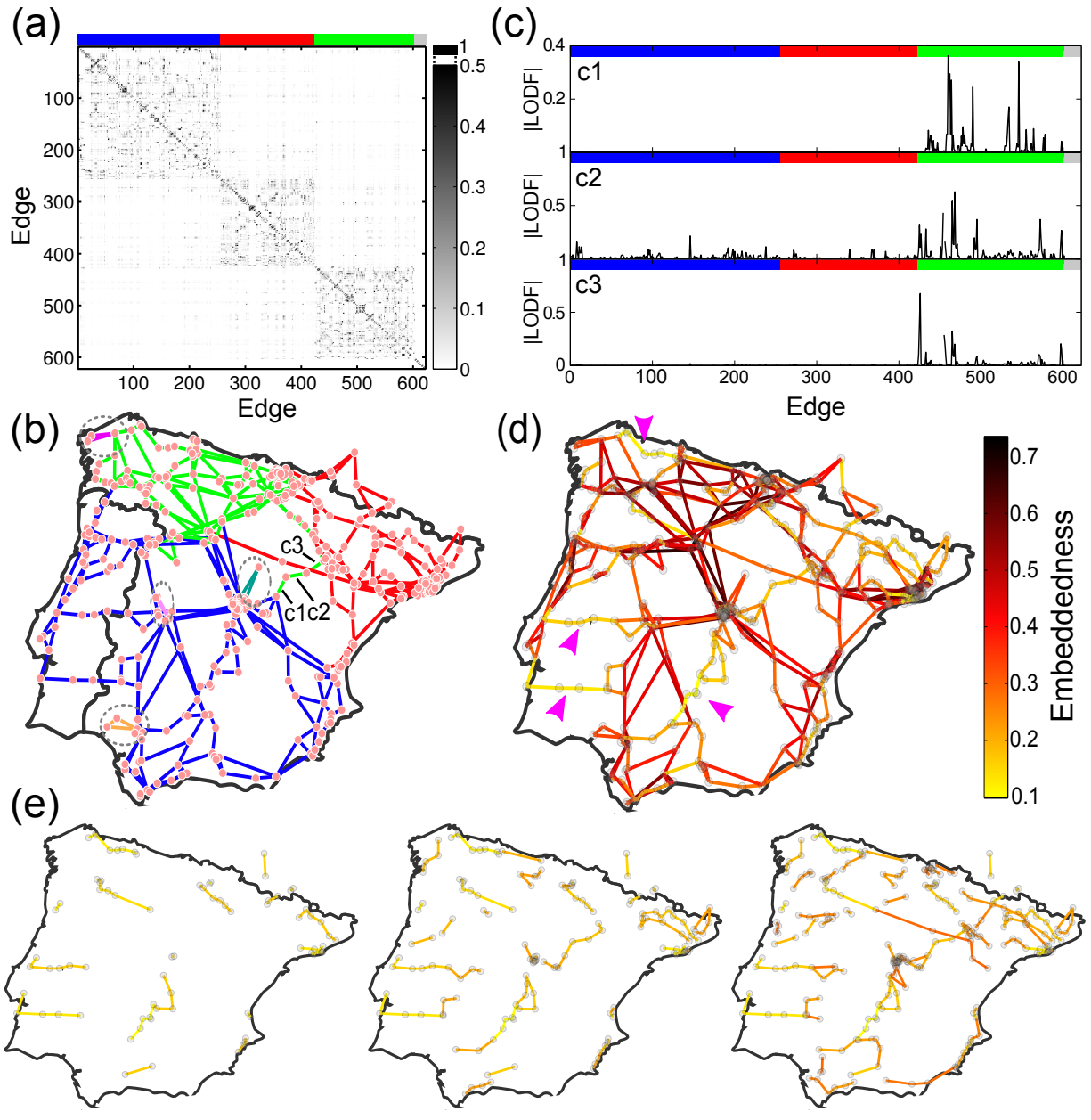
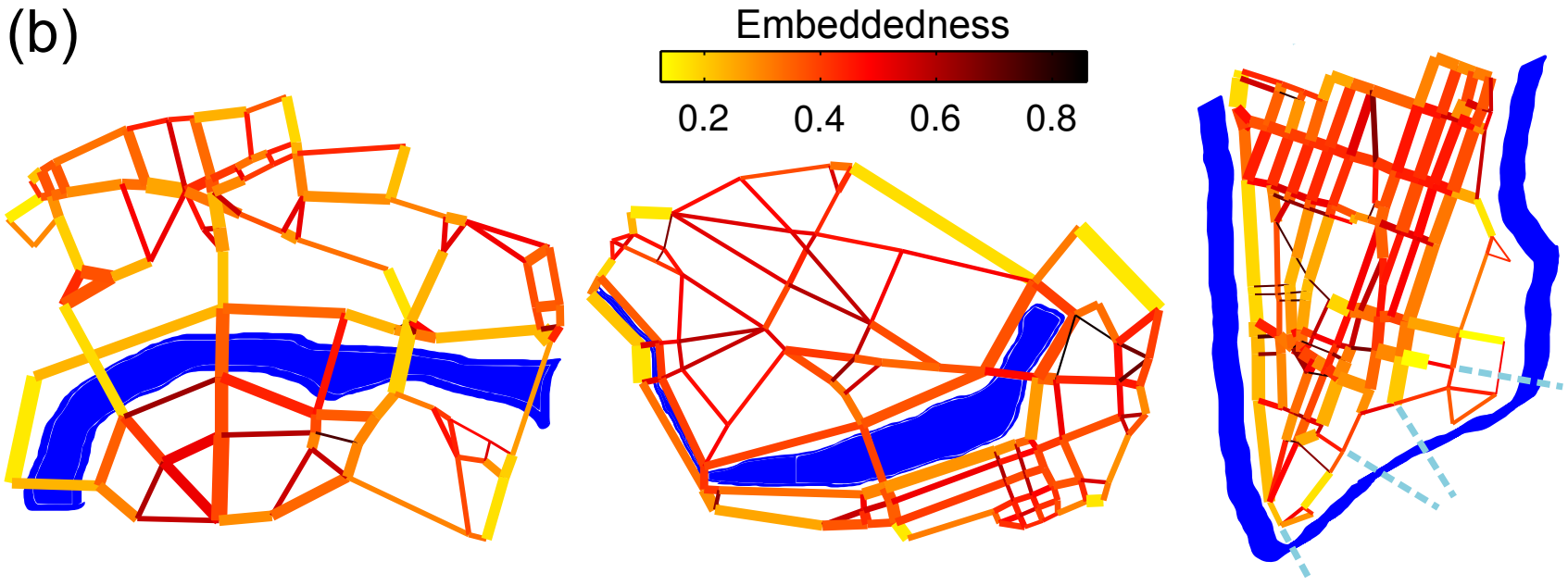


Applications 2 – Street networks



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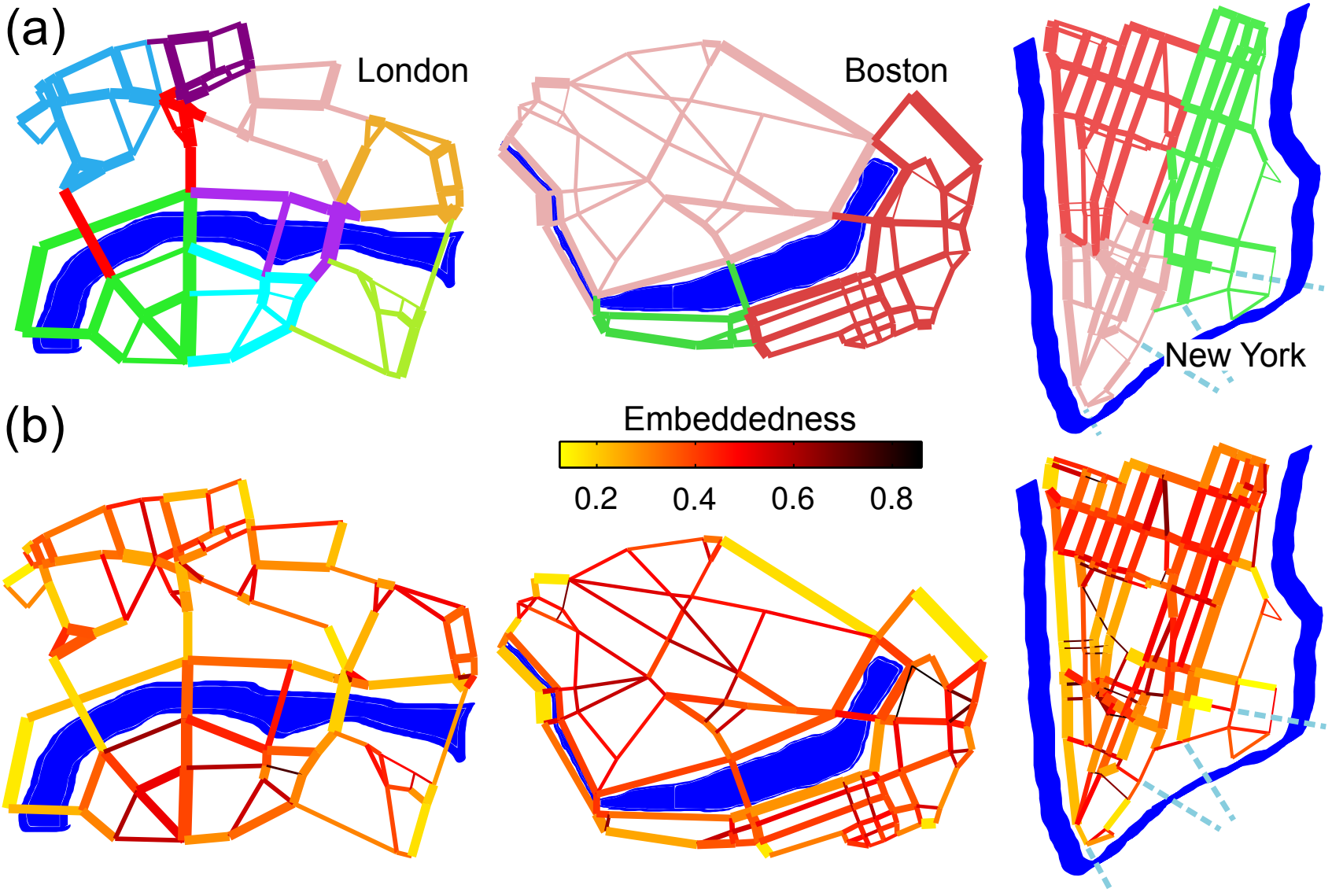
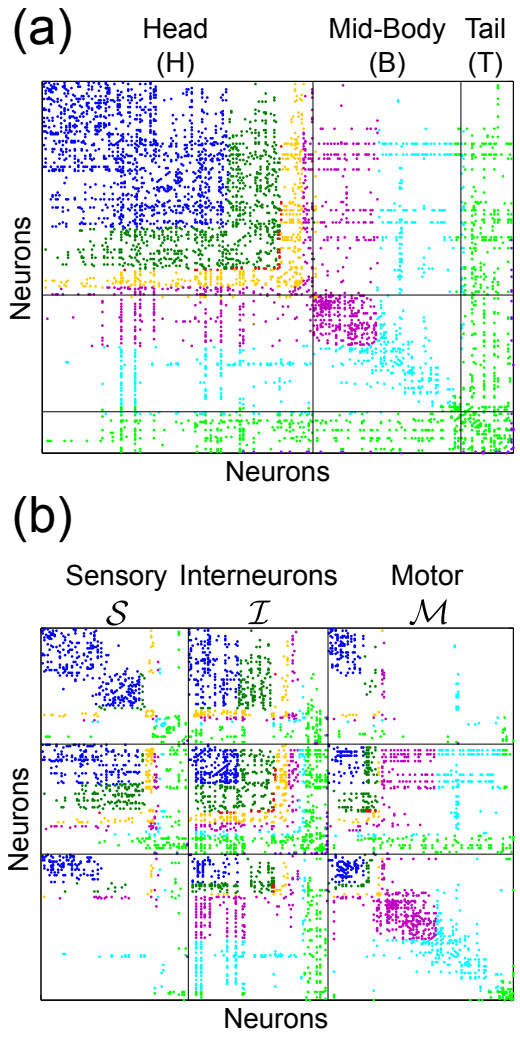


Applications 3 – C. elegans



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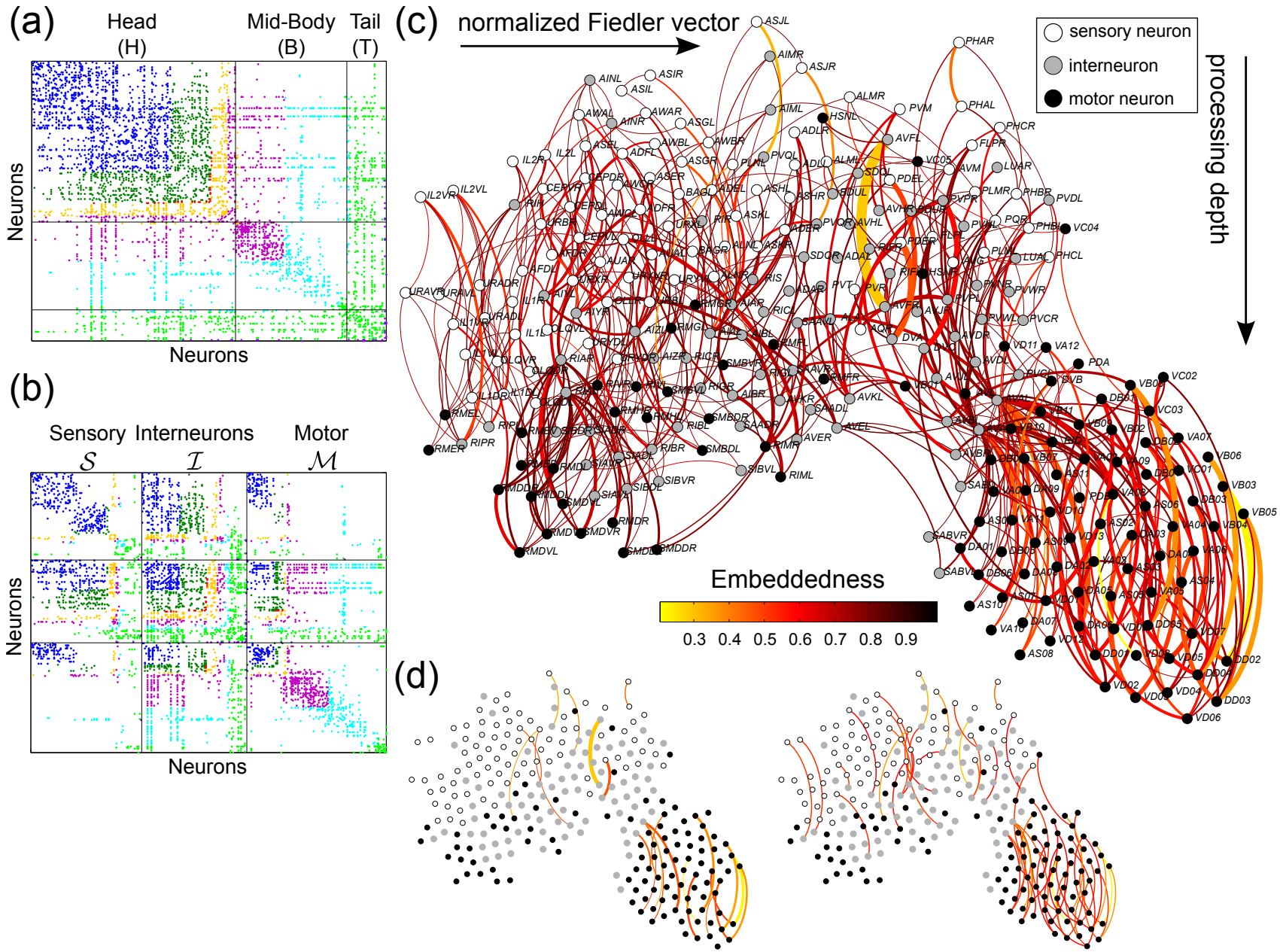


Take home messages

- Flow redistribution can characterise edge-to-edge relations
- Flow redistribution matrix – describes topological property in edge space
- Decomposable in measures with graph theoretic meaning:
  - Edge transfer function matrix (discrete Greens function)
  - Edge-embeddedness (projection into cycle space, sparsification)
- Ability to detect **non-local** effects in the edge coupling

Thank you!

The people..

- J. Lehmann (ABB)
- S. N. Yaliraki
- M. Barahona

The money...

- ONR
- EPSRC
- Studienstiftung des dt. Volkes

Everybody else

–

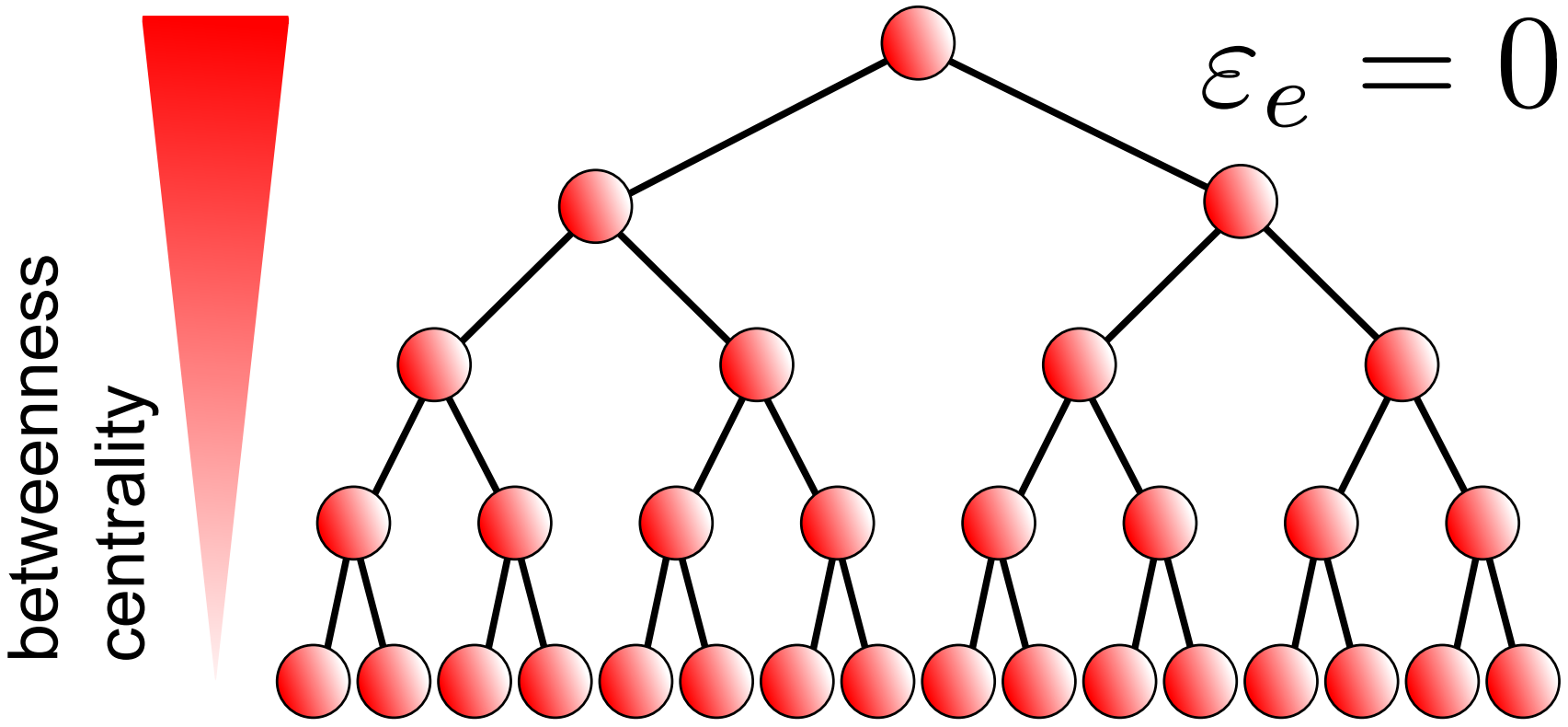
Thanks for listening!

Things left in the dark...

Schaub, M.T.; Lehmann, J.; Yaliraki, S.N. & Barahona, M., *Structure of complex networks: Quantifying edge-to-edge relations by failure-induced flow redistribution*, Network Science, April 2014, Vol. 2(1), pp. 66-89

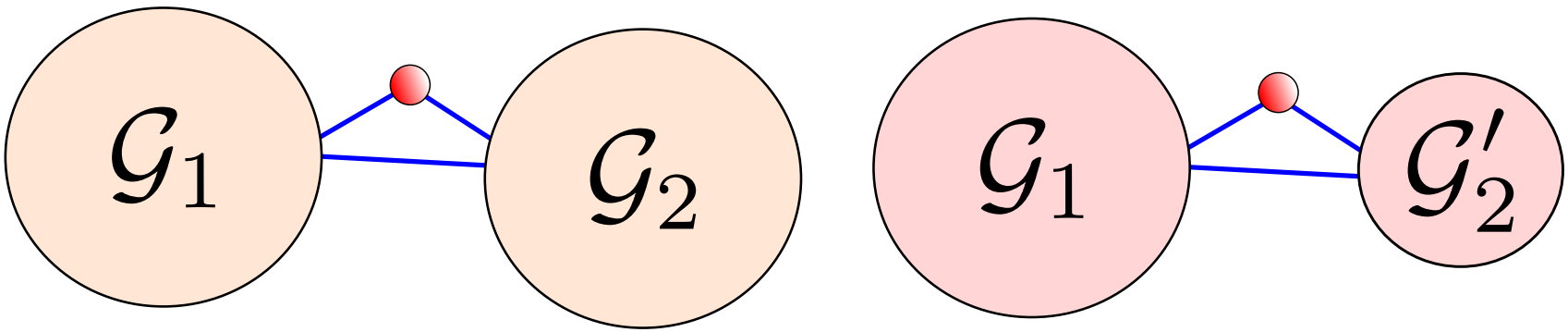
QUESTIONS?

Embeddedness vs betweenness centrality



Embeddedness vs flow betweenness centrality

flow betweeness of connecting links changes



$\mathcal{E}_e$ $\mathcal{E}_e$

Embeddedness of connecting links invariant